Available online at www.sciencedirect.com

ScienceDirect

Computers & Industrial Engineering 52 (2007) 29–40

computers & industrial engineering

www.elsevier.com/locate/dsw

Inventory and pricing strategies for deteriorating items with shortages: A discounted cash flow approach

Chung-Yuan Dye [a,*], Liang-Yuh Ouyang [b], Tsu-Pang Hsieh [b]

[a] *Department of Business Administration, Shu-Te University, Yen Chao, Kaohsiung 824, Taiwan, ROC*
[b] *Graduate Institute of Management Sciences, Tamkang University, Tamsui, Taipei 251, Taiwan, ROC*

Received 14 January 2006; received in revised form 25 October 2006; accepted 25 October 2006

Abstract

In this article, we consider an infinite horizon, single product economic order quantity where demand and deterioration rate are continuous and differentiable function of price and time, respectively. In addition, we allow for shortages and completely backlogged. The objective is to find the optimal inventory and pricing strategies maximizing the net present value of total profit over the infinite horizon. For any given selling price, we first prove that the optimal replenishment schedule not only exists but is unique. Next, we show that the total profit per unit time is a concave function of price when the replenishment schedule is given. We then provide a simple algorithm to find the optimal selling price and replenishment schedule for the proposed model. Finally, we use a couple of numerical examples to illustrate the algorithm.

Keywords: EOQ; Pricing; Time-value of money; Deterioration; Shortages

1. Introduction

In many inventory systems, the deterioration of goods is a realistic phenomenon. It is well known that certain products such as medicine, volatile liquids, blood bank, food stuff and many others, decrease under deterioration (vaporization, damage, spoilage, dryness and so on) during their normal storage period. As a result, while determining the optimal inventory policy of that type of products, the loss due to deterioration can not be ignored. In the literature of inventory theory, the deteriorating inventory models have been continually modified so as to accommodate more practical features of the real inventory systems. The analysis of deteriorating inventory began with Ghare and Schrader (1963), who established the classical no-shortage inventory model with a constant rate of decay. However, it has been empirically observed that failure and life expectancy of many items can be expressed in items of Weibull distribution. This empirical observation has prompted researchers to represent the time to deterioration of a product by a Weibull distribution. Covert and Philip (1973) extended Ghare and Schrader's (1963) model and obtained an economic order quantity model for a

* Corresponding author. Fax: +886 7 6158000x3199.
E-mail address: cydye@mail.stu.edu.tw (C.-Y. Dye).

0360-8352/$ - see front matter
doi:10.1016/j.cie.2006.10.009

variable rate of deterioration by assuming a two-parameter Weibull distribution. Researchers including Philip (1974), Misra (1975), Tadikamalla (1978), Chakrabarty, Giri, and Chaudhuri (1998) developed economic order quantity models which focused on this type of products. Therefore, a realistic model is the one that treats the deterioration rate as a time varying function. Some models have been proposed for more information, we refer the reader to Ghare and Schrader (1963) and the references therein.

Pricing is a major strategy for a seller to achieve its maximum profit. Consequently, several researchers in operations management have studied the joint lot sizing and pricing decisions for deteriorating items. Cohen (1977) jointly determined the optimal replenishment cycle and price for inventory that is subject to continuous decay over time at a constant rate. Wee (1997, 1999) extended Cohen's model to consider a Weibull distribution deterioration item with shortage. Then, Wee and Law (2001) extended Wee's (1997) model and applied the DCF (Discounted Cash Flow) approach to the finite planning horizon in which the replenishment cycle is known. All the above models assumed a linear form of the price-dependent demand rate. Recently, Hwang and Shinn (1997) addressed the joined price and lot size determination problem for an exponentially deteriorating product and iso-elastic demand when the vendor permits delay in payments. Mukhopadhyay, Mukherjee, and Chaudhuri (2004, 2005) re-established Cohen's model (1977) by taking iso-elastic demand and a varying deterioration rate. However, it is very restrictive to assume that the item deteriorates at a specific distribution and the demand follows a specific function. To relax these assumptions, Abad (1996, 2001) discussed a lot-sizing problem for a product with a general deterioration function and a general demand function, allowing shortages and partial backlogging. Unfortunately, he does not use the stockout cost (includes backorder cost and the lost sale cost) in the formulation of the objective function since these costs are not easy to estimate, and its immediate impact is that there is a lower service level to customers.

Companies have recognized that besides maximizing profit, customer satisfaction plays an important role for getting and keeping a successful position in a competitive market. The proper inventory level should be set based on the relationship between the investment in inventory and the service level. For inventory systems, the average cost approach is more frequently used by the practitioners when the discount rate is at a negligible level. However, as the time value of money is taken into account in the inventory systems, an alternative is to determine the decision variables by minimizing the discounted value of all future costs (i.e., the net present value (NPV) of total cost). Hadley (1964) compared the optimal order quantities determined by minimizing these two different objective functions. When the discount rate is excessive, he obtained the optimal reorder intervals with significant differences for these two models. Further, Rachamadugu (1988) developed error bounds for EOQ model by minimizing net present value. Since the net present value is the standard methodology in theoretical analysis and the most frequently used method for making financial decisions, we develop a generalized inventory model using the net present value of its total profit as the objective function to amend the papers of Cohen (1977), Wee (1997), Wee and Law (2001), Abad (1996, 2001) and Mukhopadhyay et al. (2004, 2005) with a view to making the model more relevant and applicable in practice. In the next section, the assumptions and notation related to this study are presented. Then, we prove that the optimal replenishment policy not only exists but is unique, for any given selling price. Next, we show that the net present value of total profit is a concave function of selling price when the replenishment schedule is given. We also provide a simple algorithm to find the optimal replenishment schedule and selling price for the proposed model. Finally, we use a couple of numerical examples to illustrate the procedure of solving the model.

2. Model notation and assumptions

To develop the mathematical model of inventory replenishment policy, the notation adopted in this paper is as below:

A = the replenishment cost per order
c = the purchasing cost per unit
s = the selling price per unit, where $s > c$
c_1 = the holding cost per unit time
c_2 = the backorder cost per unit time
r = the discount rate

Q = the ordering quantity per cycle
t_1 = the time at which the inventory level reaches zero
t_2 = the length of period during which shortages are allowed
T = the length of the inventory cycle, where $T = t_1 + t_2$
$I_1(t)$ = the level of positive inventory at time t, where $0 \leqslant t \leqslant t_1$
$I_2(t)$ = the level of negative inventory at time t, where $t_1 \leqslant t \leqslant t_1 + t_2$
$NPV(s, t_1, t_2)$ = the net present value of total profit

In addition, the following assumptions are imposed:

1. Replenishment rate is infinite, and lead time is zero.
2. The time horizon of the inventory system is infinite.
3. The demand rate, $d(s)$ is any non-negative, continuous, convex, decreasing function of the selling price in $[0, s_u]$.
4. The items deteriorate at a varying rate of deterioration $\theta(t)$, where $0 < \theta(t) \ll 1$.
5. Shortages are allowed and completely backlogged.

3. Mathematical formulation

Using above assumptions, the inventory level follows the pattern depicted in Fig. 1. To establish the total relevant profit function, we consider the following time intervals separately, $[0, t_1]$ and $[t_1, t_1 + t_2]$. During the interval $[0, t_1]$, the inventory is depleted due to the combined effects of demand and deterioration. Hence the inventory level is governed by the following differential equation:

$$\frac{\mathrm{d}I_1(t)}{\mathrm{d}t} = -d(s) - \theta(t)I_1(t), \quad 0 < t < t_1, \tag{1}$$

with the boundary condition $I_1(t_1) = 0$. Solving the differential Eq. (1), we get the inventory level as

$$I_1(t) = d(s)\mathrm{e}^{-g(t)} \int_t^{t_1} \mathrm{e}^{g(u)}\,\mathrm{d}u, \quad 0 \leqslant t \leqslant t_1, \tag{2}$$

where $g(z) = \int_0^z \theta(u)\,\mathrm{d}u$. Therefore, the present value of the holding cost for the first cycle is

$$c_1 \int_0^{t_1} \mathrm{e}^{-rt} I_1(t)\,\mathrm{d}t = c_1 d(s) \int_0^{t_1} \mathrm{e}^{-rt}\mathrm{e}^{-g(t)} \int_t^{t_1} \mathrm{e}^{g(u)}\,\mathrm{d}u\,\mathrm{d}t.$$

Furthermore, at time t_1, shortage occurs and the inventory level starts dropping below 0. During $[t_1, t_1 + t_2]$, the inventory level only depends on demand and is governed by the following differential equation:

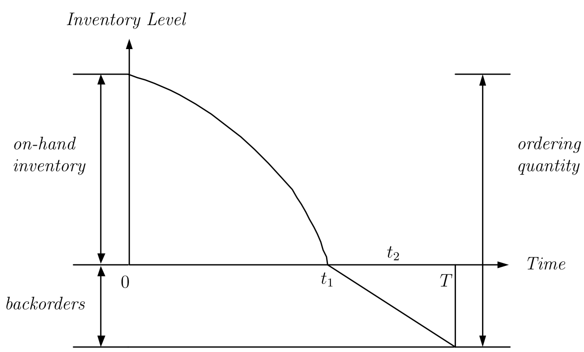

Fig. 1. Graphical representation of inventory system.

$$\frac{\mathrm{d}I_2(t)}{\mathrm{d}t} = -d(s), \quad t_1 < t < t_1 + t_2, \tag{3}$$

with the boundary condition $I_2(t_1) = 0$. Solving the differential Eq. (3), we obtain the inventory level as

$$I_2(t) = -d(s)(t - t_1), \quad t_1 \leqslant t \leqslant t_1 + t_2. \tag{4}$$

and the present value of the backorder cost for the first cycle is

$$-c_2 \int_{t_1}^{t_1+t_2} \mathrm{e}^{-rt} I_2(t)\,\mathrm{d}t = \frac{c_2 d(s)\mathrm{e}^{-r(t_1+t_2)}(\mathrm{e}^{rt_2} - rt_2 - 1)}{r^2}.$$

At time $t = t_1 + t_2$, all the shortages during the period t_2 are backordered. With an instantaneous cash transactions during sales, the present value of sales revenue for the first cycle is

$$s\left[\int_0^{t_1} \mathrm{e}^{-rt} d(s)\,\mathrm{d}t - \mathrm{e}^{-r(t_1+t_2)} I_2(t_1 + t_2)\right] = sd(s)\left[\frac{1 - \mathrm{e}^{-rt_1}}{r} + t_2\mathrm{e}^{-r(t_1+t_2)}\right].$$

Likewise, the order quantity and the present value of purchase cost can be obtained as

$$Q = I_1(0) - I_2(t_1 + t_2) = d(s)\left[\int_0^{t_1} \mathrm{e}^{g(u)}\,\mathrm{d}u + t_2\right],$$

and

$$c[I_1(0) - \mathrm{e}^{-r(t_1+t_2)} I_2(t_1 + t_2)] = cd(s)\left[\int_0^{t_1} \mathrm{e}^{g(t)}\,\mathrm{d}t + t_2\mathrm{e}^{-r(t_1+t_2)}\right],$$

respectively.

Now, we are ready to derive the present value of cash flows for the first cycle which comprises the present values of the sales revenues, replenishment cost, purchase cost, holding cost and backorder cost. After some algebraic manipulation, the present value of cash flows for the first cycle is obtained as follows:

$$\begin{aligned} TP(s, t_1, t_2) = {} & (s - c)d(s)\left[\frac{1 - \mathrm{e}^{-rt_1}}{r} + t_2\mathrm{e}^{-r(t_1+t_2)}\right] - A - cd(s)\int_0^{t_1} \left[\mathrm{e}^{g(t)} - \mathrm{e}^{-rt}\right]\mathrm{d}t \\ & - c_1 d(s)\int_0^{t_1} \mathrm{e}^{-rt}\mathrm{e}^{-g(t)}\int_t^{t_1} \mathrm{e}^{g(u)}\,\mathrm{d}u\,\mathrm{d}t - \frac{c_2 d(s)\mathrm{e}^{-r(t_1+t_2)}(\mathrm{e}^{rt_2} - rt_2 - 1)}{r^2}. \end{aligned} \tag{5}$$

Let $NPV(s, t_1, t_2)$ be the present values of total profit over horizon $[0, \infty]$. Then, we have

$$\begin{aligned} NPV(s, t_1, t_2) &= \sum_{n=0}^{\infty} TP(s, t_1, t_2)\mathrm{e}^{-nr(t_1+t_2)} = TP(s, t_1, t_2)\sum_{n=0}^{\infty} \mathrm{e}^{-nr(t_1+t_2)} \\ &= \frac{TP(s, t_1, t_2)}{1 - \mathrm{e}^{-r(t_1+t_2)}}. \end{aligned} \tag{6}$$

Now, the problem is to determine s, t_1 and t_2 such that $NPV(s, t_1, t_2)$ is maximized. To maximize the net present value of total profit, it is necessary to solve the following three equations simultaneously:

$$\begin{aligned}\frac{\partial NPV(s,t_1,t_2)}{\partial t_1} &= \frac{1}{1-\mathrm{e}^{-r(t_1+t_2)}}\Big\{(s-c)\big[\mathrm{e}^{-rt_1}-rt_2\mathrm{e}^{-r(t_1+t_2)}\big]-c\big[\mathrm{e}^{g(t_1)}-\mathrm{e}^{-rt_1}\big].\\ &\quad -c_1\mathrm{e}^{g(t_1)}\int_0^{t_1}\mathrm{e}^{-g(t)-rt}\,\mathrm{d}t+\frac{c_2\mathrm{e}^{-r(t_1+t_2)}(\mathrm{e}^{rt_2}-rt_2-1)}{r}\Big\}d(s)\\ &\quad -\frac{r\mathrm{e}^{-r(t_1+t_2)}}{\big[1-\mathrm{e}^{-r(t_1+t_2)}\big]^2}\Big\{(s-c)d(s)\Big[\frac{1-\mathrm{e}^{-rt_1}}{r}+t_2\mathrm{e}^{-r(t_1+t_2)}\Big]\\ &\quad -cd(s)\int_0^{t_1}\big[\mathrm{e}^{g(t)}-\mathrm{e}^{-rt}\big]\mathrm{d}t-c_1d(s)\int_0^{t_1}\mathrm{e}^{-rt}\mathrm{e}^{-g(t)}\int_t^{t_1}\mathrm{e}^{g(u)}\,\mathrm{d}u\,\mathrm{d}t\\ &\quad -\frac{c_2d(s)\mathrm{e}^{-r(t_1+t_2)}(\mathrm{e}^{rt_2}-rt_2-1)}{r^2}-A\Big\}=0, \end{aligned} \tag{7}$$

$$\begin{aligned}\frac{\partial NPV(s,t_1,t_2)}{\partial t_2} &= \frac{\mathrm{e}^{-r(t_1+t_2)}}{1-\mathrm{e}^{-r(t_1+t_2)}}\{(s-c)-[c_2+r(s-c)]t_2\}d(s)\\ &\quad -\frac{r\mathrm{e}^{-r(t_1+t_2)}}{\big[1-\mathrm{e}^{-r(t_1+t_2)}\big]^2}\Big\{(s-c)d(s)\Big[\frac{1-\mathrm{e}^{-rt_1}}{r}+t_2\mathrm{e}^{-r(t_1+t_2)}\Big]\\ &\quad -cd(s)\int_0^{t_1}[\mathrm{e}^{g(t)}-\mathrm{e}^{-rt}]\mathrm{d}t-c_1d(s)\int_0^{t_1}\mathrm{e}^{-rt}\mathrm{e}^{-g(t)}\int_t^{t_1}\mathrm{e}^{g(u)}\,\mathrm{d}u\,\mathrm{d}t\\ &\quad -\frac{c_2d(s)\mathrm{e}^{-r(t_1+t_2)}(\mathrm{e}^{rt_2}-rt_2-1)}{r^2}-A\Big\}=0, \end{aligned} \tag{8}$$

and

$$\begin{aligned}\frac{\partial NPV(s,t_1,t_2)}{\partial s} &= \frac{1}{1-\mathrm{e}^{-r(t_1+t_2)}}\Big\{[d(s)+(s-c)d'(s)]\Big[\frac{1-\mathrm{e}^{-rt_1}}{r}+t_2\mathrm{e}^{-r(t_1+t_2)}\Big]\\ &\quad -cd'(s)\int_0^{t_1}[\mathrm{e}^{g(t)}-\mathrm{e}^{-rt}]\mathrm{d}t-c_1d'(s)\int_0^{t_1}\mathrm{e}^{-rt}\mathrm{e}^{-g(t)}\int_t^{t_1}\mathrm{e}^{g(u)}\,\mathrm{d}u\,\mathrm{d}t\\ &\quad -\frac{c_2d'(s)\mathrm{e}^{-r(t_1+t_2)}(\mathrm{e}^{rt_2}-rt_2-1)}{r^2}\Big\}=0. \end{aligned} \tag{9}$$

After some algebraic manipulation, Eqs. (7) and (8) reduce to the following:

$$c[\mathrm{e}^{g(t_1)+rt_1}-1]+c_1\mathrm{e}^{g(t_1)+rt_1}\int_0^{t_1}\mathrm{e}^{-g(t)-rt}\,\mathrm{d}t=\frac{[c_2+r(s-c)](1-\mathrm{e}^{-rt_2})}{r}, \tag{10}$$

and

$$\begin{aligned}&[1-\mathrm{e}^{-r(t_1+t_2)}]\{(s-c)-[c_2+r(s-c)]t_2\}d(s)\\ &-r\Big\{(s-c)d(s)\Big[\frac{1-\mathrm{e}^{-rt_1}}{r}+t_2\mathrm{e}^{-r(t_1+t_2)}\Big]-cd(s)\int_0^{t_1}[\mathrm{e}^{g(t)}-\mathrm{e}^{-rt}]\mathrm{d}t\\ &-c_1d(s)\int_0^{t_1}\mathrm{e}^{-rt}\mathrm{e}^{-g(t)}\int_t^{t_1}\mathrm{e}^{g(u)}\,\mathrm{d}u\,\mathrm{d}t-\frac{c_2d(s)\mathrm{e}^{-r(t_1+t_2)}(\mathrm{e}^{rt_2}-rt_2-1)}{r^2}-A\Big\}=0. \end{aligned} \tag{11}$$

Applying Eqs. (10) and (11), we can obtain the following result.

Theorem 1. *For any given s, we have*

(a) *Eqs. (10) and (11) has a unique solution.*
(b) *The solution in (a) satisfies the second order conditions for a global maximum.*

Proof. See Appendix A for details. □

From the analysis carried out so far, we know that, for any given positive s, the point (t_1^*, t_2^*) which maximizes the net present value of total profit not only exists but also is unique.

Next, we study the condition under which the optimal selling price also exists and is unique. For any t_1^* and t_2^*, the first-order necessary condition for $NPV(s|t_1^*, t_2^*)$ to be maximum is

$$\begin{aligned}\frac{\mathrm{d}NPV(s|t_1^*,t_2^*)}{\mathrm{d}s} = {} & \frac{1}{1-\mathrm{e}^{-r(t_1^*+t_2^*)}}\left\{[d(s)+(s-c)d'(s)]\left[\frac{1-\mathrm{e}^{-rt_1^*}}{r}+t_2^*\mathrm{e}^{-r(t_1^*+t_2^*)}\right]\right.\\ & -cd'(s)\int_0^{t_1^*}\left[\mathrm{e}^{g(t)}-\mathrm{e}^{-rt}\right]\mathrm{d}t - c_1 d'(s)\int_0^{t_1^*}\mathrm{e}^{-rt}\mathrm{e}^{-g(t)}\int_t^{t_1^*}\mathrm{e}^{g(u)}\,\mathrm{d}u\,\mathrm{d}t\\ & \left.-\frac{c_2 d'(s)\mathrm{e}^{-r(t_1^*+t_2^*)}(\mathrm{e}^{rt_2^*}-rt_2^*-1)}{r^2}\right\}=0, \end{aligned} \tag{12}$$

where $d'(s)$ is the derivative of $d(s)$ with respect to s. We then obtain following result.

Theorem 2. *For any given values of t_1 and t_2, if the gross profit, $(s-c)d(s)$, is concave (i.e., $\mathrm{d}\{(s-c)d(s)\}/\mathrm{d}s$ is an decreasing function of s), then there exists a unique s^* which maximizes $NPV(s|t_1,t_2)$.*

Proof. Taking the second-order derivative of $NPV(s|t_1,t_2)$ with respect to s, we obtain

$$\begin{aligned}\frac{\mathrm{d}^2NPV(s|t_1,t_2)}{\mathrm{d}s^2} = {} & \frac{1}{1-\mathrm{e}^{-r(t_1+t_2)}}\left\{[2d'(s)+(s-c)d''(s)]\left[\frac{1-\mathrm{e}^{-rt_1}}{r}+t_2\mathrm{e}^{-r(t_1+t_2)}\right]\right.\\ & -cd''(s)\int_0^{t_1}\left[\mathrm{e}^{g(t)}-\mathrm{e}^{-rt}\right]\mathrm{d}t - c_1 d''(s)\int_0^{t_1}\mathrm{e}^{-rt}\mathrm{e}^{-g(t)}\int_t^{t_1}\mathrm{e}^{g(u)}\,\mathrm{d}u\,\mathrm{d}t\\ & \left.-\frac{c_2 d''(s)\mathrm{e}^{-r(t_1+t_2)}(\mathrm{e}^{rt_2}-rt_2-1)}{r^2}\right\}, \end{aligned} \tag{13}$$

where $d''(s)$ is the second-order derivative of $d(s)$ with respect to s. Since

$$\frac{\mathrm{d}^2(s-c)d(s)}{\mathrm{d}s^2} = 2d'(s)+(s-c)d''(s) < 0 \text{ and } \frac{\mathrm{d}^2d(s)}{\mathrm{d}s^2} > 0,$$

it is clear that $\mathrm{d}^2NPV(s|t_1,t_2)/\mathrm{d}s^2 < 0$. Consequently, $NPV(s|t_1,t_2)$ is a strictly concave function of s. Therefore, there exists a unique value of s that maximizes $NPV(s|t_1,t_2)$. This completes the proof. □

Because $d'(s) < 0$, it is clear from Eq. (12) that $\mathrm{d}NPV(s|t_1^*,t_2^*)/\mathrm{d}s = 0$ has a solution if and only if $d(s)+(s-c)d'(s) < 0$. The solution of $d(s)+(s-c)d'(s) = 0$, say s_1, is the lower bound for the optimal selling price s^* such that $\mathrm{d}NPV(s|t_1^*,t_2^*)/\mathrm{d}s = 0$. Note that if gross profit is an increasing function of s, then price and gross profit will always move in the same direction, hence the retailer can realize infinite gross profit by setting an infinite s. It is impossible.

Combining Theorems 1 and 2, we propose the following algorithm for solving the problem.

Algorithm.

Step 1 Start with $j = 0$ and the initial trial value of $s_j = s_1$, which is a solution to $d(s)+(s-c)d'(s) = 0$.
Step 2 Find the optimal replenishment schedule, t_1^* and t_2^* from Eqs. (10) and (11), for a given selling price s_j.
Step 3 Use the result in Step 2, and then determine the optimal s_{j+1} by Eq. (12).
Step 4 If the difference between s_j and s_{j+1} is sufficiently small, set $s^* = s_{j+1}$, then (s^*, t_1^*, t_2^*) is the optimal solution and stop. Otherwise, set $j = j+1$ and go back to Step 2.

4. Numerical example

To illustrate the results, let us apply the proposed algorithm to solve the following numerical examples. The results can be found by applying the subrouting "FindRoot" in Mathematica version 4.1.

Example 1. We first redo the same example of Wee and Law (2001) to see the optimal replenishment policy while considering time value of money. $A = 80$, $c = 5$, $c_1 = 0.6$, $c_2 = 1.4$, $r = 0.08$, $d(s) = 200 - 4s$, where $s \in [0, 50]$ and $\theta(t) = \alpha \times \beta \times t^{\beta-1} = 0.05 \times 1.5 \times t^{0.5}$ (i.e., Weibull deterioration rate, where α is scale parameter and β is shape parameter). Solving $d(s) + (s - c)d'(s) = 0$ first, we get $s_l = s_0 = 27.5$. Then, applying the algorithm, after 3 iterations, the optimal values of s, t_1 and t_2 are $s^* = 27.7533$, $t_1^* = 0.9638$ and $t_2^* = 0.4043$, respectively. The ordering quantity and net present value of total profit obtained here is $Q^* = 123.4$ and $NPV^* = 23861.0$, respectively.

Compare with Wee and Law's (2001) model, and choose the same value for the planning horizon, H, we find that the $H/(t_1^* + t_2^*) = 10/(1.3681) = 7.3094$. Then the optimal number of cycle (denoted by N) is 8 or the total number of order is 9. Over the finite horizon $[0, H]$, since $N(t_1^* + t_2^*) = 8 \times (0.9638 + 0.4043) = 10.9448 > 10$, we adjust per replenishment cycle by the following policy:

$$t_1' = \frac{H}{N} \frac{t_1^*}{t_1^* + t_2^*}, \quad t_2' = \frac{H}{N} \frac{t_2^*}{t_1^* + t_2^*} \quad \text{and} \quad T' = t_1' + t_2'.$$

Further, by using the adjustment policy of the time schedules above and Eq. (18) in Wee and Law (2001), we can obtain the total net present-value profit over horizon $[0, H]$ by the following approximation:

$$TP(s^*, t_1', t_2') \times \frac{1 - e^{-rH}}{1 - e^{-rT'}} - A \times e^{-rH}.$$

In Table 1, we summarize a comparative study of the results of Wee and Law (2001) and those of the present model. Even though the approximate profit in Wee and Law (2001) is overrated with neglecting the second- and higher-order terms of such as the scale parameter of Weibull distribution deterioration and the discount rate, we see that our model presents the improved profit. Furthermore, if the planning horizon is given, then the solution obtained here is a good approximation to estimate the optimal number of replenishments to avoid using a brute force enumeration.

Example 2. We then use the some parameters given in Mukhopadhyay et al. (2004). $A = 250$, $c = 40$, $d(s) = 16 \times 10^7 \times s^{-3.21}$, where $s \in [0, 75]$ and $\theta(t) = 0.05 \times 2 \times t^{2-1} = 0.1t$. Besides, we take $c_1 = 4.50$, $c_2 = 5.00$ and $r = 0.08$. By solving $d(s) + (s - c)d'(s) = 0$, we obtain $s_l = s_0 = 58.0995$. After 4 iterations, we have $s^* = 59.5891$, $t_1^* = 0.2832$, $t_2^* = 0.3667$, $Q^* = 208.4$ and $NPV^* = 68{,}831.5$.

5. Conclusion

In this paper, the use of NPV as the objective function for the generalized inventory system is developed. When interest rates are high, the decision based on the average profit will be inferior to the decision based on the NPV because it ignores the discount rate – the time value of money. The analytical formulations of the problem on the general framework have been given. Furthermore, in contrast to Wee (1997, 1999), Wee and Law (2001), and Mukhopadhyay et al. (2004, 2005), the approach in this

Table 1
Summary of the comparison between the models

	Number of cycle, N	Selling price, s	The time interval			Ordering quantity per cycle, Q	Total net present-value profit
			t_1	t_2	T		
Wee and Law (2001)	8	27.7340	0.8940	0.3560	1.250	112.676	13,069.937
This model	8	27.7533	0.8806	0.3694	1.250	112.546	13,100.140

paper provides solutions better than those obtained by using Taylor Series approximation. We can also see that any deterioration rate can be applied to this model such as the three-parameter Weibull deterioration rate (e.g., Philip, 1974) and Gamma deterioration rate (e.g., Tadikamalla, 1978). Hence, the utilization of general price-dependent demand and deterioration rates make the scope of the application broader.

The proposed model can be extended in several ways. First, we can easily extend the backlogging rate of unsatisfied demand to any decreasing function $\beta(x)$, where x is the waiting time up to the next replenishment, and $0 \leqslant \beta(x) \leqslant 1$ with $\beta(0) = 1$. Second, we can also incorporate the quantity discount, and the learning curve phenomenon into the model.

Acknowledgements

The authors thank the editor and anonymous reviewers for their valuable and constructive comments, which have led to a significant improvement in the manuscript. This research was partially supported by the National Science Council of the Republic of China under Grant NSC-94-2213-E-366-006.

Appendix A. The proof of Theorem 1

(a) From Eq. (10), we know the LHS of Eq. (10) is a strictly increasing function in t_1. Thus, for given any $t_1 \in [0, \widehat{t_1}]$, t_2 can be uniquely determined as a function of t_1, that is

$$t_2 = -\frac{1}{r} \ln \left\{ 1 - \frac{r\left\{ c[\mathrm{e}^{g(t_1)+rt_1} - 1] + c_1 \mathrm{e}^{g(t_1)+rt_1} \int_0^{t_1} \mathrm{e}^{-g(t)-rt}\,\mathrm{d}t \right\}}{c_2 + r(s-c)} \right\}, \tag{14}$$

where $\widehat{t_1}$ is the root of the following equation

$$\frac{c_2 + r(s-c)}{r} = c\left[\mathrm{e}^{g(\widehat{t_1})+r\widehat{t_1}} - 1\right] + c_1 \mathrm{e}^{g(\widehat{t_1})+r\widehat{t_1}} \int_0^{\widehat{t_1}} \mathrm{e}^{-g(t)-rt}\,\mathrm{d}t.$$

Next, in order to prove the existence of the solution, from Eq. (10), by taking the implicit differentiation with respect to t_1, it gets

$$[c_2 + r(s-c)]\mathrm{e}^{-rt_2} \frac{\mathrm{d}t_2}{\mathrm{d}t_1} = \left[c + c_1 \int_0^{t_1} \mathrm{e}^{-g(t)-rt}\,\mathrm{d}t\right][\theta(t_1) + r]\mathrm{e}^{g(t_1)+rt_1} + c_1, \tag{15}$$

and hence $\mathrm{d}t_2/\mathrm{d}t_1 > 0$.

From Eq. (11), for notational convenience, let

$$\begin{aligned} G(t_1) &= [1 - \mathrm{e}^{-r(t_1+t_2)}]\{(s-c) - [c_2 + r(s-c)]t_2\}d(s) \\ &\quad - r\left\{ (s-c)d(s)\left[\frac{1-\mathrm{e}^{-rt_1}}{r} + t_2 \mathrm{e}^{-r(t_1+t_2)}\right] - cd(s)\int_0^{t_1} [\mathrm{e}^{g(t)} - \mathrm{e}^{-rt}]\mathrm{d}t \right. \\ &\quad \left. - c_1 d(s) \int_0^{t_1} \mathrm{e}^{-rt}\mathrm{e}^{-g(t)} \int_t^{t_1} \mathrm{e}^{g(u)}\,\mathrm{d}u\,\mathrm{d}t - \frac{c_2 d(s)\mathrm{e}^{-r(t_1+t_2)}(\mathrm{e}^{rt_2} - rt_2 - 1)}{r^2} - A \right\}. \end{aligned}$$

After assembling Eqs. (10) and (15), the first derivative of $G(t_1)$ with respect to t_1 becomes

$$
\begin{aligned}
\frac{\mathrm{d}G(t_1)}{\mathrm{d}t_1} = {} & r\mathrm{e}^{-r(t_1+t_2)}\{(s-c)-[c_2+r(s-c)]t_2\}d(s)\left(1+\frac{\mathrm{d}t_2}{\mathrm{d}t_1}\right) \\
& -[1-\mathrm{e}^{-r(t_1+t_2)}][c_2+r(s-c)]d(s)\frac{\mathrm{d}t_2}{\mathrm{d}t_1} \\
& -r\Bigg\{\frac{[c_2+r(s-c)]d(s)\mathrm{e}^{-rt_1}(1-\mathrm{e}^{-rt_2})}{r} \\
& -c[\mathrm{e}^{g(t_1)}-\mathrm{e}^{-rt_1}]d(s)-c_1\mathrm{e}^{g(t_1)}d(s)\int_0^{t_1}\mathrm{e}^{-g(t)-rt}\,\mathrm{d}t \\
& +\frac{c_2 d(s)\mathrm{e}^{-r(t_1+t_2)}(\mathrm{e}^{rt_2}-rt_2-1)}{r}\left(1+\frac{\mathrm{d}t_2}{\mathrm{d}t_1}\right) \\
& +(s-c)d(s)\mathrm{e}^{-r(t_1+t_2)}(1-rt_2)\left(1+\frac{\mathrm{d}t_2}{\mathrm{d}t_1}\right) \\
& -\frac{c_2 d(s)\mathrm{e}^{-r(t_1+t_2)}(\mathrm{e}^{rt_2}-1)}{r}\left(1+\frac{\mathrm{d}t_2}{\mathrm{d}t_1}\right)\Bigg\} \\
= {} & -[1-\mathrm{e}^{-r(t_1+t_2)}][c_2+r(s-c)]d(s)\frac{\mathrm{d}t_2}{\mathrm{d}t_1}<0.
\end{aligned}
$$

Thus, $G(t_1)$ is a strictly decreasing function of $t_1\in[0,\widehat{t}_1]$.

Further, because $G(0)=rA>0$ and

$$
\begin{aligned}
\lim_{t_1\to\widehat{t}_1^-} G(t_1) = {} & \lim_{t_1\to\widehat{t}_1^-}[1-\mathrm{e}^{-r(t_1+t_2)}]\{(s-c)-[c_2+r(s-c)]t_2\}d(s) \\
& -\lim_{t_1\to\widehat{t}_1^-} r\Bigg\{(s-c)d(s)\left[\frac{1-\mathrm{e}^{-rt_1}}{r}+t_2\mathrm{e}^{-r(t_1+t_2)}\right]-cd(s)\int_0^{t_1}[\mathrm{e}^{g(t)}-\mathrm{e}^{-rt}]\mathrm{d}t \\
& -c_1 d(s)\int_0^{t_1}\mathrm{e}^{-rt}\mathrm{e}^{-g(t)}\int_t^{t_1}\mathrm{e}^{g(u)}\,\mathrm{d}u\,\mathrm{d}t-\frac{c_2 d(s)\mathrm{e}^{-r(t_1+t_2)}(\mathrm{e}^{rt_2}-rt_2-1)}{r^2}-A\Bigg\} \\
= {} & (s-c)d(s)-\lim_{t_2\to\infty}[1-\mathrm{e}^{-r(t_1+t_2)}]\{[c_2+r(s-c)]t_2\}d(s) \\
& -r\Bigg\{(s-c)d(s)\left[\frac{1-\mathrm{e}^{-r\widehat{t}_1}}{r}\right]-cd(s)\int_0^{\widehat{t}_1}[\mathrm{e}^{g(t)}-\mathrm{e}^{-rt}]\mathrm{d}t \\
& -c_1 d(s)\int_0^{\widehat{t}_1}\mathrm{e}^{-rt}\mathrm{e}^{-g(t)}\int_t^{\widehat{t}_1}\mathrm{e}^{g(u)}\,\mathrm{d}u\,\mathrm{d}t-A\Bigg\}=-\infty<0,
\end{aligned}
$$

the Intermediate Value Theorem implies that the root of $G(t_1)=0$ is unique. This completes the proof. □

(b) Since $NPV(s,t_1,t_2)=TP(s,t_1,t_2)/[1-\mathrm{e}^{-r(t_1+t_2)}]$, we know that the necessary conditions for maximum are

$$
\frac{\partial NPV(s,t_1,t_2)}{\partial t_1}=\frac{-r\mathrm{e}^{-r(t_1+t_2)}TP(s,t_1,t_2)}{[1-\mathrm{e}^{-r(t_1+t_2)}]^2}+\frac{1}{1-\mathrm{e}^{-r(t_1+t_2)}}\frac{\partial TP(s,t_1,t_2)}{\partial t_1}=0,
$$

and

$$
\frac{\partial NPV(s,t_1,t_2)}{\partial t_2}=\frac{-r\mathrm{e}^{-r(t_1+t_2)}TP(s,t_1,t_2)}{[1-\mathrm{e}^{-r(t_1+t_2)}]^2}+\frac{1}{1-\mathrm{e}^{-r(t_1+t_2)}}\frac{\partial TP(s,t_1,t_2)}{\partial t_2}=0.
$$

From Theorem 1(a), we have shown that the optimal solution for Eqs. (10) and (11) is unique, we obtain that

$$\left.\frac{\partial TP(s,t_1,t_2)}{\partial t_1}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} = \left.\frac{\partial TP(s,t_1,t_2)}{\partial t_2}\right|_{(t_1,t_2)=(t_1^*,t_2^*)}.$$

Using the result, we have

$$\begin{aligned}
\left.\frac{\partial^2 NPV(s,t_1,t_2)}{\partial t_1^2}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} &= \left.\frac{r^2 \mathrm{e}^{-r(t_1+t_2)} TP(s,t_1,t_2)}{[1-\mathrm{e}^{-r(t_1+t_2)}]^2}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} \\
&\quad + \frac{1}{1-\mathrm{e}^{-r(t_1+t_2)}} \left.\frac{\partial^2 TP(s,t_1,t_2)}{\partial t_1^2}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} \\
&= \frac{r}{1-\mathrm{e}^{-r(t_1+t_2)}} \left.\frac{\partial TP(s,t_1,t_2)}{\partial t_1}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} \\
&\quad + \frac{1}{1-\mathrm{e}^{-r(t_1+t_2)}} \left.\frac{\partial^2 TP(s,t_1,t_2)}{\partial t_1^2}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} \\
&= \left.\frac{d(s)\mathrm{e}^{-r(t_1+t_2)}}{1-\mathrm{e}^{-r(t_1+t_2)}} \{r(s-c) - r[c_2 + r(s-c)]t_2\}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} \\
&\quad + \frac{d(s)\mathrm{e}^{-r(t_1+t_2)}}{1-\mathrm{e}^{-r(t_1+t_2)}} \left\{ -[c_2 + r(s-c)] \frac{\mathrm{d}t_2}{\mathrm{d}t_1} \right. \\
&\quad \left.\left. + r[c_2 + r(s-c)]t_2 - r(s-c) \right\}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} \\
&= -\frac{d(s)[c_2 + r(s-c)]\mathrm{e}^{-r(t_1+t_2)}}{1-\mathrm{e}^{-r(t_1+t_2)}} \left.\frac{\mathrm{d}t_2}{\mathrm{d}t_1}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} < 0,
\end{aligned}$$

$$\begin{aligned}
\left.\frac{\partial^2 NPV(s,t_1,t_2)}{\partial t_2^2}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} &= \left.\frac{r^2 \mathrm{e}^{-r(t_1+t_2)} TP(s,t_1,t_2)}{[1-\mathrm{e}^{-r(t_1+t_2)}]^2}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} \\
&\quad + \frac{1}{1-\mathrm{e}^{-r(t_1+t_2)}} \left.\frac{\partial^2 TP(s,t_1,t_2)}{\partial t_2^2}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} \\
&= \frac{r}{1-\mathrm{e}^{-r(t_1+t_2)}} \left.\frac{\partial TP(s,t_1,t_2)}{\partial t_2}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} \\
&\quad + \frac{1}{1-\mathrm{e}^{-r(t_1+t_2)}} \left.\frac{\partial^2 TP(s,t_1,t_2)}{\partial t_2^2}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} \\
&= \left.\frac{rd(s)\mathrm{e}^{-r(t_1+t_2)}}{1-\mathrm{e}^{-r(t_1+t_2)}} \{(s-c) - [c_2 + r(s-c)]t_2\}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} \\
&\quad + \frac{d(s)\mathrm{e}^{-r(t_1+t_2)}}{1-\mathrm{e}^{-r(t_1+t_2)}} \{-[c_2 + r(s-c)] \\
&\quad - r\{(s-c) - [c_2 + r(s-c)]t_2\}\}|_{(t_1,t_2)=(t_1^*,t_2^*)} \\
&= -\left.\frac{d(s)[c_2 + r(s-c)]\mathrm{e}^{-r(t_1+t_2)}}{1-\mathrm{e}^{-r(t_1+t_2)}}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} < 0,
\end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial^2 NPV(s,t_1,t_2)}{\partial t_1 \partial t_2} &= \frac{r^2 \mathrm{e}^{-r(t_1+t_2)} TP(s,t_1,t_2)}{[1-\mathrm{e}^{-r(t_1+t_2)}]^2}\bigg|_{(t_1,t_2)=(t_1^*,t_2^*)} \\
&\quad + \frac{1}{1-\mathrm{e}^{-r(t_1+t_2)}} \frac{\partial^2 TP(s,t_1,t_2)}{\partial t_2 \partial t_1}\bigg|_{(t_1,t_2)=(t_1^*,t_2^*)} \\
&= \frac{r}{1-\mathrm{e}^{-r(t_1+t_2)}} \frac{\partial TP(s,t_1,t_2)}{\partial t_1}\bigg|_{(t_1,t_2)=(t_1^*,t_2^*)} \\
&\quad + \frac{1}{1-\mathrm{e}^{-r(t_1+t_2)}} \frac{\partial^2 TP(s,t_1,t_2)}{\partial t_2 \partial t_1}\bigg|_{(t_1,t_2)=(t_1^*,t_2^*)} \\
&= \frac{rd(s)\mathrm{e}^{-r(t_1+t_2)}}{1-\mathrm{e}^{-r(t_1+t_2)}} \{(s-c) - [c_2 + r(s-c)]t_2\}\bigg|_{(t_1,t_2)=(t_1^*,t_2^*)} \\
&\quad - \frac{rd(s)\mathrm{e}^{-r(t_1+t_2)}}{1-\mathrm{e}^{-r(t_1+t_2)}} \{(s-c) - [c_2 + r(s-c)]t_2\}\bigg|_{(t_1,t_2)=(t_1^*,t_2^*)} = 0.
\end{aligned}
$$

Thus, the determinant of Hessian matrix $\mathbf{H}$ at the stationary point (t_1^*, t_2^*) is

$$
\begin{aligned}
\det(\mathbf{H}) &= \frac{\partial^2 NPV(s,t_1,t_2)}{\partial t_1^2}\bigg|_{(t_1,t_2)=(t_1^*,t_2^*)} \times \frac{\partial^2 NPV(s,t_1,t_2)}{\partial t_2^2}\bigg|_{(t_1,t_2)=(t_1^*,t_2^*)} \\
&\quad - \frac{\partial^2 NPV(s,t_1,t_2)}{\partial t_1 \partial t_2}\bigg|_{(t_1,t_2)=(t_1^*,t_2^*)} \times \frac{\partial^2 NPV(s,t_1,t_2)}{\partial t_1 \partial t_2}\bigg|_{(t_1,t_2)=(t_1^*,t_2^*)}. \\
&= \frac{d(s)[c_2 + r(s-c)]\mathrm{e}^{-r(t_1+t_2)}}{1-\mathrm{e}^{-r(t_1+t_2)}} \frac{\mathrm{d}t_2}{\mathrm{d}t_1}\bigg|_{(t_1,t_2)=(t_1^*,t_2^*)} \\
&\quad \times \frac{d(s)[c_2 + r(s-c)]\mathrm{e}^{-r(t_1+t_2)}}{1-\mathrm{e}^{-r(t_1+t_2)}}\bigg|_{(t_1,t_2)=(t_1^*,t_2^*)} \\
&= \left\{\frac{d(s)[c_2 + r(s-c)]\mathrm{e}^{-r(t_1+t_2)}}{1-\mathrm{e}^{-r(t_1+t_2)}}\right\}^2 \frac{\mathrm{d}t_2}{\mathrm{d}t_1}\bigg|_{(t_1,t_2)=(t_1^*,t_2^*)} > 0.
\end{aligned}
$$

Hence, the Hessian matrix $\mathbf{H}$ at point (t_1^*, t_2^*) is negative definite. Consequently, we can conclude that the stationary point (t_1^*, t_2^*) is a global maximum for our optimization problem. This completes the proof. □

References

Abad, P. L. (1996). Optimal pricing and lot sizing under conditions of perishability and partial backordering. *Management Science, 42*, 1093–1104.

Abad, P. L. (2001). Optimal price and order size for a reseller under partial backordering. *Computers & Operations Research, 28*, 53–65.

Chakrabarty, T., Giri, B. C., & Chaudhuri, K. S. (1998). An EOQ model for items with weibull distribution deterioration, shortages and trended demand: an extension of Philip's model. *Computers & Operations Research, 25*, 649–657.

Cohen, M. A. (1977). Joint pricing and ordering policy for exponentially decaying inventory with known demand. *Naval Research Logistics Quarterly, 24*, 257–268.

Covert, R. B., & Philip, G. S. (1973). An EOQ model with Weibull distribution deterioration. *AIIE Transactions, 5*, 323–326.

Ghare, P. M., & Schrader, G. H. (1963). A model for exponentially decaying inventory system. *Journal of Industrial Engineering, 14*, 238–243.

Hadley, G. (1964). A comparison of order quantities computed using the average annual cost and the discounted cost. *Management Science, 10*, 472–476.

Hwang, H., & Shinn, S. W. (1997). Retailer's pricing and lot sizing policy for exponentially deteriorating products under the condition of permissible delay in payments. *Computers & Operations Research, 24*, 539–547.

Misra, R. B. (1975). Optimum production lot size model for a system with deteriorating inventory. *International Journal of Production Research, 13*, 495–505.

Mukhopadhyay, S., Mukherjee, R. N., & Chaudhuri, K. S. (2004). Joint pricing and ordering policy for a deteriorating inventory. *Computers and Industrial Engineering, 47*, 339–349.

Mukhopadhyay, S., Mukherjee, R. N., & Chaudhuri, K. S. (2005). An EOQ model with two-parameter Weibull distribution deterioration and price-dependent demand. *International Journal of Mathematical Education in Science and Technology, 36*, 25–33.

Philip, G. C. (1974). A generalized EOQ model for items with Weibull distribution deterioration. *AIIE Transactions, 6*, 159–162.

Rachamadugu, R. (1988). Error bounds for EOQ. *Naval Research Logistics, 35*, 419–435.

Tadikamalla, P. R. (1978). An EOQ inventory model for items with gamma distribution. *AIIE Transactions, 10*, 108–112.

Wee, H. M. (1997). A replenishment policy for items with a price-dependent demand and a varying rate of deterioration. *Production Planning & Control, 8*, 494–499.

Wee, H. M. (1999). Deteriorating inventory model with quantity discount, pricing and partial backordering. *International Journal of Production Economics, 59*, 511–518.

Wee, H. M., & Law, S. T. (2001). Replenishment and pricing policy for deteriorating items taking into account the time-value of money. *International Journal of Production Economics, 71*, 213–220.